Journal of Engineering Mathematics, Vol. 6, No. 1, January 1972
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

Two Shift Theorems Leading to Lower Bounds to Eigenvalues

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(Received April 16, 1971 and in revised form June 25, 1971)

SUMMARY

Two theorems are proved by which the "other bound" can be obtained from the Rayleigh quotient, i.e., the bound which cannot be obtained by the Rayleigh–Ritz method.

The method presented here utilizes a "shift operation" which is the redistribution of weight functions in the Rayleigh quotient.

1. Introduction

One of the most powerful tools in the calculation of eigenvalues is the Rayleigh–Ritz method. A more elaborate one is the Galerkin method, which is based on the same philosophy as the Rayleigh–Ritz method. A drawback of both methods is that they supply just one bound: An upper bound for minimum formulation and a lower bound for maximum variational formulation.

Attempts were made to obtain the "other" bound. The Weinstein method can be applied in some cases. A method which yields the "other bound" is given in Ref. [1]; however, this method yields a bound which is not close to the correct value. A systematic approach for a particular set of problems is given in Ref. [2], which is a particular case of the Denominator Shift theorem proved in this work.

The contribution of this work is the extension and generalization of the method of Ref. [2]. This work shows how to obtain lower bounds to eigenvalues for a more general class of problems: The considered operators are more general and may appear in both numerator and denominator of the variational formulations.

2. Analysis

Let an eigenvalue problem have a variational formulation

$$\lambda^2 = \min \left\{ \int_D A(f)dv \Big/ \int_D B(f)dv \right\} \tag{1}$$

with a set of boundary conditions. dv, of course, is of the same dimensions as D. A and B are operators, having the general form

$$\left. \begin{aligned} A(f) &= \sum_i^n \alpha_i a_i(f) \\ B(f) &= \sum_j^n \beta_j b_j(f) \end{aligned} \right\} \tag{2}$$

where a_i and b_i are operators, and α_i and β_i are given functions.

The solution of Eq. (1) is f_1.

Without loss of generality both numerator and denominator in Eq. (1) are assumed positive.

Let a *shift operation* be defined: Two functions ϕ and ψ are the shift of each other if:

$$\int_D \phi\, dv = \int_D \psi\, dv\,. \tag{3}$$

A result of Eq. (3) is

$$\left.\begin{aligned} & D=\bar{D}+\underline{D} \\ & \phi>\psi \text{ in } \bar{D} \\ & \phi<\psi \text{ in } \underline{D} \\ & \int_{\bar{D}}(\phi-\psi)dv=\int_{\underline{D}}(\psi-\phi)dv \geqq 0 \end{aligned}\right\} \tag{4}$$

Let new notations be introduced in Eq. (2): One of the terms in both A and B is considered particularly, and its subscript is dropped; also, new operators, G and H, are defined:

$$\left.\begin{aligned} A(f) &= \sum_{i=1}^{k-1} \alpha_i a_i(f)+\alpha a(f)+\sum_{i=k+1}^{n} \alpha_i a_i(f) \\ B(f) &= \sum_{j=1}^{l-1} \beta_j b_j(f)+\beta b(f)+\sum_{j=l+1}^{m} \beta_j b_j(f) \\ A(f) &= G(f)+\alpha a(f) \\ B(f) &= H(f)+\beta b(f). \end{aligned}\right\} \tag{5}$$

3. The Denominator Shift Theorem

(This theorem is a generalization of a theorem given in Ref. [2], where $H(f)=0$). Let β and δ be the shift of each other:

$$\left.\begin{aligned} & \int_D \beta\, dv=\int_D \delta\, dv \\ & D=\bar{D}+\underline{D} \\ & \beta>\delta \text{ in } \bar{D} \\ & \beta<\delta \text{ in } \underline{D} \\ & \int_{\bar{D}}(\beta-\delta)dv=\int_{\underline{D}}(\delta-\beta)dv>0\,. \end{aligned}\right\} \tag{6}$$

Let the maximum of $b(f_1)$ in $\bar{D}$ be $\bar{b}$, and the minimum of $b(f_1)$ in $\underline{D}$ be $\underline{b}$, i.e.,

$$\begin{aligned} & \bar{b} \geqq b(f_1) \text{ in } \bar{D} \\ & \underline{b} \leqq b(f_1) \text{ in } \underline{D}\,. \end{aligned} \tag{7}$$

Let another eigenvalue problem be (see Eq. (5))

$$\mu^2=\min\left\{\int_D A(f)dv \Big/ \int_D [H(f)+\delta b(f)]dv\right\} \tag{8}$$

with the same boundary conditions as Eq. (1). Now, if

$$\underline{b}>\bar{b} \tag{9}$$

then, *Theorem*:

$$\mu^2<\lambda^2 \tag{10}$$

Proof:

$$\begin{aligned} \int_{\bar{D}} B(f_1)dv &= \int_{\bar{D}}[H(f_1)+\beta b(f_1)]dv = \\ &= \int_{\bar{D}}[H(f_1)+\delta b(f_1)]dv+\int_{\bar{D}}(\beta-\delta)b(f_1)dv \\ &\leqq \int_{\bar{D}}[H(f_1)+\delta b(f_1)]dv+\bar{b}\int_{\bar{D}}(\beta-\delta)dv\,. \end{aligned} \tag{11}$$

$$\int_{\underline{D}} B(f_1)dv = \int_{\underline{D}} [H(f_1)+\beta b(f_1)]dv =$$

$$= \int_{\underline{D}} [H(f_1)+\delta b(f_1)]dv - \int_{\underline{D}} (\delta-\beta)b(f_1)dv$$

$$\leqq \int_{\underline{D}} [H(f_1)+\delta b(f_1)]dv - \underline{b} \int_{\underline{D}} (\delta-\beta)dv\,. \tag{12}$$

Addition of Eq. (11) and Eq. (12) yield

$$\int_D B(f_1)dv \leqq \int_D [H(f_1)+\delta b(f_1)]dv - (\underline{b}-\bar{b}) \int_{\bar{D}} (\beta-\delta)dv$$

$$\leqq \int_D [H(f_1)+\delta b(f_1)]dv\,. \tag{13}$$

From Eqs. (1) and (8):

$$\lambda^2 = \min\left\{ \int_D A(f)dv \bigg/ \int_D B(f)dv \right\} =$$

$$= \left\{ \int_D A(f_1)dv \bigg/ \int_D B(f_1)dv \right\}$$

$$\leqq \left\{ \int_D A(f_1)dv \bigg/ \int_D [H(f_1)+\delta b(f_1)]dv \right\}$$

$$\leqq \min\left\{ \int_{\bar{D}} A(f)dv \bigg/ \int_{\bar{D}} [H(f)+\delta b(f)]dv \right\} = \mu^2 \qquad \text{Q.E.D.}$$

4. The Numerator Shift Theorem

Let α and γ be the shift of each other:

$$\begin{aligned}
&\int_D \alpha\, dv = \int_D \gamma\, dv \\
&D = \bar{D} + D \\
&\alpha > \gamma \text{ in } \bar{D} \\
&\alpha < \gamma \text{ in } \underline{D} \\
&\int_{\bar{D}} (\alpha-\gamma)dv = \int_{\bar{D}} (\gamma-\alpha)dv \geqq 0\,.
\end{aligned} \tag{14}$$

Let the minimum of $a(f_1)$ in $\bar{D}$ be $\bar{a}$, and the maximum of $a(f_1)$ in $\underline{D}$ be $\underline{a}$, i.e.,

$$\begin{aligned}
&\bar{a} \leqq a(f_1) \text{ in } \bar{D} \\
&\underline{a} \geqq a(f_1) \text{ in } \underline{D}\,.
\end{aligned} \tag{15}$$

Let another eigenvalue problem be (see Eq. (5))

$$\nu^2 = \min\left\{ \int_D [G(f)+\gamma a(f)]dv \bigg/ \int_D B(f)dv \right\} \tag{16}$$

with the same boundary conditions as Eq. (1). Now, if

$$\bar{a} > \underline{a} \tag{17}$$

then, *Theorem*:

$$\nu^2 < \lambda^2 \tag{18}$$

Proof: .

$$\int_{\bar{D}} A(f_1)\,dv = \int_{\bar{D}} [G(f_1)+\alpha a(f_1)]\,dv =$$

$$= \int_{\bar{D}} [G(f_1)+\gamma a(f_1)\,dv + \int_{\bar{D}} (\alpha-\gamma)a(f_1)\,dv$$

$$\geqq \int_{\bar{D}} [G(f_1)+\gamma a(f_1)]\,dv + \bar{a}\int_{\bar{D}} (\alpha-\gamma)\,dv\,. \tag{19}$$

$$\int_{\underline{D}} A(f_1)\,dv = \int_{\bar{D}} [G(f_1)+\alpha a(f_1)]\,dv =$$

$$= \int_{\underline{D}} [G(f_1)+\gamma a(f_1)]\,dv - \int_{\underline{D}} (\gamma-\alpha)a(f_1)\,dv$$

$$\geqq \int_{\underline{D}} [G(f_1)+\gamma a(f_1)]\,dv - \underline{a}\int_{\underline{D}} (\gamma-\alpha)\,dv\,. \tag{20}$$

Addition of Eq. (19) and Eq. (20) yields

$$\int_D A(f_1)\,dv \geqq \int_D [G(f_1)+\gamma a(f_1)]\,dv + (\bar{a}-\underline{a})\int_{\bar{D}} (\alpha-\gamma)\,dv$$

$$\geqq \int_D [G(f_1)+\gamma a(f_1)]\,dv\,. \tag{21}$$

From Eq. (1) and Eq. (16)

$$\lambda^2 = \min\left\{\int_D A(f)\,dv \Big/ \int_D B(f)\,dv\right\} =$$

$$= \left\{\int_D A(f_1)\,dv \Big/ \int_D B(f_1)\,dv\right\}$$

$$\geqq \left\{\int_D [G(f_1)+\gamma a(f_1)]\,dv \Big/ \int_D B(f_1)\,dv\right\}$$

$$\geqq \min\left\{\int_D [G(f)+\gamma a(f)]\,dv \Big/ \int_D B(f)\,dv\right\} = v^2 \qquad \text{Q.E.D.} \tag{22}$$

5. Corollaries

1. Both numerator and denominator shifts may be used simultaneously, with different divisions of D into $\bar{D}$ and $\underline{D}$, for each shift.
2. Several shifts may be used in the denominator, with different division of D into $\bar{D}$ and $\underline{D}$ for each shift and each term in $\Sigma\,\beta_i a_i(f)$.
3. Several shifts may be used in the numerator, with different division of D into $\bar{D}$ and $\underline{D}$ for each shift and each term in $\Sigma\,\beta_i a_i(f)$.
4. The functions γ and δ need only to be continuous by parts and integrable.

The practical use of the theorems consists of the following steps:
(a) Identification of the general character of the solution in the sense of Eq. (9) and Eq. (17), e.g., that the solution or the relevant part of the operator is larger in certain known subregions than in another subregion (see argument leading to Eq. (26) in the numerical example).
(b) The invention of a new function (γ, δ in theorems, S^2 in the numerical example), such that the resulting Euler–Lagrange equation is easily and exactly solved.
(c) Scaling of the new weight function to be the shift of the old one (see Eq. (26)).
Once these steps are followed the sought bound is directly obtained.

6. Numerical Example

This example is artificially constructed to demonstrate the use of the theorems.

Consider a string of variable mass: Eq. (23):

$$\left.\begin{array}{l} y''+\lambda^2(1+\varepsilon x^2)y=0 \\ y=0 \text{ at } x=\pm\pi/2\,. \end{array}\right\} \tag{23}$$

The equivalent variational formulation is

$$\lambda^2 = \min\left\{\int_{-\pi/2}^{\pi/2}(y')^2\,dx \Big/ \int_{-\pi/2}^{\pi/2}(1+\varepsilon x^2)y^2\,dx\right\}. \tag{24}$$

By the standard Rayleigh–Ritz method, assuming $y=\cos x$

$$\lambda^2 \leqq \frac{\pi/2}{\pi/2+\varepsilon(\pi^3/24-\pi/4)} = \frac{1}{1+\varepsilon(\pi^2/12-\frac{1}{2})}. \tag{25}$$

Clearly, y has its maximum at $x=0$ and decreases monotonously toward $x=\pm\pi/2$. Therefore, the condition of Eq. (7) is satisfied by shifting the weight function $(1+\varepsilon x^2)$ toward smaller x-values. Denote

$$S^2 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1+\varepsilon x^2)dx = 1+\varepsilon\frac{\pi^2}{12}. \tag{26}$$

The shifted* problem, in a variational formulation is

$$\Lambda^2 = \min\left\{\int_{-\pi/2}^{\pi/2}(y')^2\,dx \Big/ \int_{-\pi/2}^{\pi/2} S^2 y^2\,dx\right\} \tag{27}$$

with the Euler–Lagrange equation

$$y''+\Lambda^2 S^2 y = 0 \tag{28}$$

and with the solution

$$y=\cos(\Lambda S x)\,, \qquad \Lambda = 1/S\,. \tag{29}$$

Thus, by the denominator shift theorem

$$\lambda^2 \geqq \left(1+\varepsilon\frac{\pi^2}{12}\right)^{-1}. \tag{30}$$

Thus, from Eqs. (25) and (30)

$$\left[1+\varepsilon\left(\frac{\pi^2}{12}-\frac{1}{2}\right)\right]^{-1} > \lambda^2 > \left(1+\varepsilon\frac{\pi^2}{12}\right)^{-1}. \tag{31}$$

REFERENCES

[1] D. Pnueli, A computation scheme for the asymptotic Nusselt Number in ducts of arbitrary cross-section, *International Journal of Heat and Mass Transfer*, 10 (1967) 1743.

[2] D. Pnueli, Lower bounds to the eigenvalues in one-dimensional problems by a shift in the weight function, *Journal of Applied Mechanics, ASME*, 37, E, 2 (1970) 267.

* The region D: $-\pi/2<x<\pi/2$, is divided into $\bar{D}$: $(-\pi/2<x<-\pi/2\sqrt{3},\ \pi/2\sqrt{3}<x<\pi/2)$ where $(1+\varepsilon x^2)>S^2$, and into $\underline{D}$: $-\pi/2\sqrt{3}<x<\pi/2\sqrt{3}$ where $1+\varepsilon x^2<S^2$.